Strong Coupling Expansion of
the $SU(3)$ and $U(3)$ Effective Actions

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A set of programs is developed to calculate algebraically the effective action for a gauge system with fermions and arbitrary external sources on a lattice. In particular a strong coupling series for QCD is derived.

0. INTRODUCTION

This paper is a sequel to [1], where a strong coupling expansion was developed for the free energy of lattice gauge systems with the standard one-plaquette interaction including fermions. This investigation is part of a program aimed at making explicit calculations in quantum chromodynamics in the strong coupling region. Extrapolation schemes then have to be used to translate these results to the weak coupling, physical region. The basic method, the effective action method, was developed in [2, 3]. It uses the Euclidean version of the action on a four-dimensional lattice. The gauge fields are integrated out of the generating functional and the remaining integral is rewritten in terms of elementary meson- and baryon-fields. The object of the present paper and [1] is to calculate the effective action thus obtained as a series in the parameter $\beta = 1/g^2$, where g is the gauge coupling constant. In [1] the terms of the free energy (logarithm of the generating functional) in a certain order in β were expressed in terms of the one-link free energy. In the present paper specific choices for the gauge group are made ($SU(3)$ or $U(3)$). This allows the free energy functional to be obtained in terms of the (elementary) meson and baryon fields.

In Section 1 the structure of the calculation is described, and in Section 2 technical details are discussed. In Section 3 the implementation on a computer and the results concerning the effective action are discussed. In Section 4 the effective potential is calculated as a sample application. The effective potential determines the vacuum expectation value of the meson-field in the saddle point approximation. The saddle point approximation consists of expanding the fields around their vacuum expectation values and is motivated by the large N limit. The meson part of the action is proportional to N in this limit, which exhibits the parameter $1/N$. Also numerical results substantiate the approximation for $N = 3$, cf. [2, 3]. This remains true for the full action, including baryons.

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In general, the aim is to calculate the coefficients of β^n as accurately as needed. Section 5 contains the conclusions.

1. METHOD

The generating functional for n-point functions using the lattice regularisation and Euclidean action is given by [5]

$$Z(J, \bar{\eta}, \eta) = \int D\bar{\psi} \, D\psi \, DU \exp \left\{ S_\psi + S_U + \sum_x (N \operatorname{tr} JM + \bar{\eta}\beta + \bar{\beta}\eta) \right\}, \quad (1.1)$$

where

$$S_\psi = \sum_{x,\mu} \bar{\psi}(x) \, P_\mu^- \, U(x,\mu) \, \psi(x + a^\mu) + \bar{\psi}(x + a^\mu) \, P_\mu^+ \, U^+(x,\mu) \, \psi(x)$$

$$- \sum_x \bar{\psi}(x) \, M\psi(x) \tag{1.2}$$

and

$$S_U = \beta \sum_{\mathscr{P}} \operatorname{Tr} U_{\mathscr{P}} \equiv \sum_{x,\mu,v \neq \mu} \operatorname{Tr} U_{(x,\mu,v)}, \tag{1.3a}$$

$$U_{(x,\mu,v)} = U(x,\mu) \, U(x + a^\mu, v) \, U^+(x + a^v, \mu) \, U^+(x, v). \tag{1.3b}$$

Here $U(x,\mu)$ is a gauge matrix U_b^a sitting on link $(x, x + a^\mu)$ and to keep the notation in agreement with the previous papers $1/g^2$ is denoted by β in contrast to the more common convention $1/g^2 = \beta/2N$. We will absorb the mass M in the source J. Furthermore

$$P_\mu^\pm = \tfrac{1}{2}(r \pm \gamma_\mu), \qquad 0 \leqslant r \leqslant 1. \tag{1.4}$$

For $r = 1$ the fermions are Wilson fermions [5], whereas the case $r = 0$ corresponds to Susskind fermions, cf. [2, 3]. The composite meson- and baryon-fields are defined by

$$N\mathscr{M}_\beta^\alpha(x) = \psi^{a\alpha}(x) \, \bar{\psi}_{\alpha\beta}(x), \tag{1.5a}$$

$$N!\mathscr{B}^{\alpha_1 \cdots \alpha_N}(x) = \varepsilon_{a_1 \dots a_N} \psi^{a_1\alpha_1}(x) \cdots \psi^{a_N\alpha_N}(x), \tag{1.5b}$$

$$N!\bar{\mathscr{B}}_{\alpha_1 \dots \alpha_N}(x) = \varepsilon^{a_1 \cdots a_N} \bar{\psi}_{a_N\alpha_N}(x) \cdots \bar{\psi}_{a_1\alpha_1}(x), \tag{1.5c}$$

where $a = 1,..., N$ are color indices and $\alpha, \beta = 1,..., n$ are noncolor (flavor and Dirac) indices. By defining A_L and $\bar{A}_L$ on a link $L = (x, x + a^\mu)$ as

$$\bar{A}_L^b{}_a = \bar{\psi}(x)_{a\alpha} P_\mu^{-\,\alpha}{}_\beta \psi(x + a^\mu)^{b\beta}, \tag{1.6a}$$

$$A_L^b{}_a = \bar{\psi}(x + a^\mu)_{a\alpha} P_\mu^{+\,\alpha}{}_\beta \psi(x)^{b\beta}, \tag{1.6b}$$

the generating functional may be written as

$$Z(J, \bar{\eta}, \eta) = \int D\bar{\psi} \, D\psi \, DU \, \exp \left\{ S_U + \sum_L \text{tr}(\bar{A}_L U_L + A_L U_L^+) + \text{source terms} \right\}. \quad (1.7)$$

First we want to perform the gauge field integration yielding a strong coupling series, $\beta = 1/g^2 \to 0$, for the free energy functional

$$W(\beta, A, \bar{A}) = \log \int DU \, \exp \left\{ \beta \sum_{\mathscr{P}} \text{Tr} \, U_{\mathscr{P}} + \sum_L \text{Tr}(\bar{A}_L U_L + A_L U_L^+) \right\}. \quad (1.8)$$

The subsequent integration over the fermions and the transformation from composite to elementary meson- and baryon-fields was dealt with in [2, 3].

In [1] the series (1.8) was obtained up to β^4 for arbitrary sources A_L and $\bar{A}_L$. The coefficients of the various powers of β were obtained as a sum of terms, each a product of derivatives of the one-link free energy

$$W_L = \log \int DU_L \, \exp\{\text{tr}(\bar{A}_L U_L + A_L U_L^+)\}. \quad (1.9)$$

One such term will be called a *W*-bank in the following.

The one-link free energy is expressible in terms of color-neutral combinations of the A_L and $\bar{A}_L$. The number and type of independent color-neutral combinations depends on the gauge-group involved so at this stage a specific choice has to be made. For *SU*(3) the combinations chosen in [6] are

$$\kappa = \text{Tr}(A\bar{A}), \qquad \lambda = \text{Tr}(A\bar{A})^2, \, \mu = \det A, \, \bar{\mu} = \det \bar{A}. \quad (1.10)$$

For *SU*(*N*) a suitable choice would be $\mu, \bar{\mu}$ and

$$\text{Tr}(A\bar{A})^p, \, p = 1, ..., N - 1. \quad (1.11)$$

Exact expressions are known for W_L, see references cited in [1]. We use an expansion of W_L in terms of the invariants (1.10)

$$W_L = \sum_{i,j,k,l=0}^{\infty} c_{ijkl} \mu^i \bar{\mu}^j \kappa^k \lambda^l. \quad (1.12)$$

The coefficients c_{ijkl} can be found in [6].

The derivatives of W_L needed for substitution in the *W*-banks are

$$\frac{\partial^{m+n} W_L(A, \bar{A})}{\partial A^{a_1}{}_{a_2} \cdots \partial A^{a_{2m-1}}{}_{a_{2m}} \partial \bar{A}^{a_{2m+1}}{}_{a_{2m+2}} \cdots \partial \bar{A}^{a_{2m+2n-1}}{}_{a_{2m+2n}}}, \quad (1.13)$$

and we need these in a form that makes every color index explicit but leaves derivatives of W_I with respect to the invariant combinations (1.11) implicit, i.e.,

$$\sum \text{coefficient} * (\text{derivative of } W_I \text{ w.r.t. invariants})$$

$$* \prod (\text{derivatives of invariants w.r.t. } A \text{ and } \bar{A}). \qquad (1.14)$$

The derivatives of invariants with respect to A and $\bar{A}$ give colored objects of a limited number of types. For $SU(3)$ there are 13:

$$\delta^a_b, \ \varepsilon^{abc}, \ \varepsilon_{abc}, \ A^a{}_b, \ \bar{A}^a{}_b, \ (A\bar{A})^a{}_b, \ (\bar{A}A)^a{}_b,$$

$$(A\bar{A}A)^a{}_b, \ (\bar{A}A\bar{A})^a{}_b, \ \varepsilon^{abp}\varepsilon_{cdq}A^p{}_q, \ \varepsilon^{abp}\varepsilon_{cdq}\bar{A}^p{}_q,$$

$$\varepsilon^{apq}\varepsilon_{brs}A^r{}_p A^s{}_q, \ \varepsilon^{apq}\varepsilon_{brs}\bar{A}^r{}_p \bar{A}^s{}_q. \qquad (1.15)$$

Contraction of color indices and replacement of A and $\bar{A}$ by their expression (1.6) in terms of fermion fields then gives the terms of (1.14) in terms of (composite) meson- and baryon-fields (1.5). In an intermediate stage of the calculation the concept of radicals is used. A radical is defined with reference to a string of neighbouring sites $q = r_0, r_1, r_2, ..., r_k, r_{k+1} = p$ to be

$$R^a{}_b(p,q)(s) = \bar{\psi}_{b\beta_0}(q) \, P^{\beta_0}{}_{\alpha_1} N\mathscr{M}(r_1)^{\alpha_1}{}_{\beta_1} \, P^{\beta_1}{}_{\alpha_2} N\mathscr{M}(r_2)^{\alpha_2}{}_{\beta_2}$$

$$\times \cdots N\mathscr{M}(r_k)^{\alpha_k}{}_{\beta_k} P^{\beta_k}{}_{\alpha_{k+1}} \psi^{a\alpha_{k+1}}(p), \qquad (1.16)$$

where $s = (r_1, r_2, ..., r_k)$. The α and β represent both Dirac and flavor indices; the P's are unity in flavor space and in dirac space they are given by

$$P = P^+_\mu = \tfrac{1}{2}(r + \gamma_\mu) \quad \text{if} \quad r_{i+1} = r_i - a^\mu,$$

$$= P^-_\mu = \tfrac{1}{2}(r - \gamma_\mu) \quad \text{if} \quad r_{i+1} = r_i + a^\mu. \qquad (1.17)$$

Simple radicals are

$$\bar{A}^a{}_b(x, x + a^\mu) = R^a{}_b(x + a^\mu, x)(\ ), \qquad (1.18a)$$

$$A^a{}_b(x, x + a^\mu) = R^a{}_b(x, x + a^\mu)(\ ). \qquad (1.18b)$$

The combination rule for two radical is

$$R^a{}_b(p,q)(s_1) \, R^b{}_c(q,r)(s_2) = R^a{}_c(p,r)(s_2, q, s_1). \qquad (1.19)$$

In the final result of the calculation only colorless objects can occur. Thus radicals must be either self contracted, (giving rise to meson loops) or saturated by ε's (giving rise to baryon–antibaryon combinations connected by strings of mesons). A combination with one baryon and one antibaryon is formed by

$$\varepsilon_{a_1 a_2 a_3}(p) \, R^{a_1}{}_{b_1}(p,q)(s_1) \, R^{a_2}{}_{b_2}(p,q)(s_2) \, R^{a_3}{}_{b_3}(p,q)(s_3) \, \varepsilon^{b_1 b_2 b_3}(q) \qquad (1.20)$$

$$= N! \, \bar{\mathscr{B}}_{\beta_1 \beta_2 \beta_3}(q) \, H^{\beta_1}{}_{\alpha_1}(p,q)(s_1) \, H^{\beta_2}{}_{\alpha_2}(p,q)(s_2) \, H^{\beta_3}{}_{\alpha_3}(p,q)(s_3) \, N! \, \mathscr{B}^{\alpha_1 \alpha_2 \alpha_3}(p),$$

where $N = 3$ and

$$H^{\beta}{}_{\alpha}(p,q)(s) = [q]\, P^{\beta}{}_{\gamma_1} N \mathscr{M}(r_1)^{\gamma_1}{}_{\delta_1} P^{\delta_1}{}_{\gamma_2} N \mathscr{M}(r_2)^{\gamma_2}{}_{\delta_2} \cdots N \mathscr{M}(r_k)^{\gamma_k}{}_{\delta_k} P^{\delta_k}{}_{\alpha}[p] \qquad (1.21)$$

in which $s = (r_1,...,r_k)$ and the p and q in square brackets are added because they are needed to determine the P's by (1.17). Self contraction of a radical leads to a mesonloop,

$$R^a{}_a(p,p)(r_1,...,r_k) = -N \mathscr{M}(p)^{\alpha_{k+1}}{}_{\beta_0} P^{\beta_0}{}_{\alpha_1} N \mathscr{M}(r_1)^{\alpha_1}{}_{\beta_1}$$
$$\times \cdots N \mathscr{M}(r_k)^{\alpha_k}{}_{\beta_k} P^{\beta_k}{}_{\alpha_{k+1}}. \qquad (1.22)$$

Thus the form of terms of (1.14) is established as a product of

    (i)   a coefficient,

    (ii)   a product of derivatives of the one-link free energy with respect to the invariants (1.11),

    (iii)   a product of mesonloops (1.22); baryon diagrams (1.21) and the generalization thereof to diagrams containing more than one baryon–antibaryon pair.

For some applications all terms may be relevant, as for instance, for investigation of chiral invariance of the effective potential, and for calculations involving Susskind fermions ($r = 0$ in (1.4)).

But for calculations involving Wilson fermions ($r = 1$) in the saddle point approximation, use can be made of the fact that combinations $P^{\epsilon}_{\mu} \mathscr{M} P^{-\epsilon}_{\mu}$, $\epsilon = +$ or $-$, vanish if the vacuum expectation value $\mathscr{M} = v\mathbb{1}$ is substituted for the mesons. In the Wilson case P^+_{μ} and P^-_{μ} are orthogonal projectors. Such a combination $P^{\epsilon}_{\mu} \mathscr{M} P \bar{\mu}^{\epsilon}$ will now be called a "backtracker."

In calculating the meson propagators in the saddle point approximation for Wilson fermions, terms with more than two backtrackers cannot contribute and for meson four-point functions the same is true for terms with more than four backtrackers.

Most of what follows is independent of the value of r in (1.4) but all examples will refer to the case $r = 1$.

Within the effective action framework a choice can be made concerning the form of some higher order terms. A local combination of a baryon and an antibaryon can always be rewritten in terms of meson fields, because of the identity

$$\varepsilon^{abc}\varepsilon_{pqr} = \delta^a{}_p(\delta^b{}_q\delta^c{}_r - \delta^b{}_r\delta^c{}_q) + \delta^a{}_q(\delta^b{}_r\delta^c{}_p - \delta^b{}_p\delta^c{}_r)$$
$$+ \delta^a{}_r(\delta^b{}_p\delta^c{}_q - \delta^b{}_q\delta^c{}_p). \qquad (1.23)$$

Or differently stated, the three fields $\mathscr{M}(p)$, $\mathscr{B}(p)$, and $\bar{\mathscr{B}}(p)$ needed to describe color invariant combinations of $\psi^a(p)$ and $\bar{\psi}_b(p)$ are linearly independent but do have higher order algebraic relations. The single site contributions of $\mathscr{B}\bar{\mathscr{B}}$ pairs are important and we choose to rewrite them in terms of meson fields.

Furthermore the form of (1.14) has to be refined to make the baryon content of a
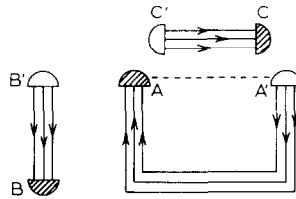
FIG. 1.   Configuration that leads to ambiguous contributions to the baryon propagator.

term explicit. This is due to the fact that factors μ or $\bar{\mu}$ may appear implicitly on some links and since, for $L = (x, x + a^v)$

$$\mu_L = N! \, \bar{\mathscr{B}}_{\alpha_1 \cdots \alpha_N}(x + a^v) \, P_v^{+\alpha_1}{}_{\beta_1} \cdots P_v^{+\alpha_N}{}_{\beta_N} \, \mathscr{B}(x)^{\beta_1 \cdots \beta_N} \qquad (1.24)$$

such a μ may change the baryon content of a term. If the term contained no explicit baryons the content will be changed from 0 to 1; if the term contained one baryon-antibaryon pair the content may change from 1 to 0 or to 2. A product $\mu\bar{\mu}$ on a link can (and will) be expressed in meson fields only. The baryon content of a term can be made explicit by expanding each derivative of the one-link free energy (w.r.t. invariants) in the number of uncompensated μ or $\bar{\mu}$'s. Thus, if g is such a derivative, expand it as

$$g(\mu, \bar{\mu}, \kappa, \lambda) = g_0(\mu\bar{\mu}, \kappa, \lambda) + \mu g_1(\mu\bar{\mu}, \kappa, \lambda) + \bar{\mu}\bar{g}_1(\mu\bar{\mu}, \kappa, \lambda)$$
$$+ \mu^2 g_2(\mu\bar{\mu}, \kappa, \lambda) + \bar{\mu}^2 \bar{g}_2(\mu\bar{\mu}, \kappa, \lambda) + \cdots . \qquad (1.25)$$

Ambiguities in the saddle point value may arise after uncompensated μ's have been added. Of course there are no ambiguities once the fermion fields are integrated out exactly, but the saddle point value will depend a little on the way the uncompensated μ's are treated. Consider, for example, the term depicted in Fig. 1. There CC' and BB' are uncompensated μ or $\bar{\mu}$'s. Now there are two possible choices to eliminate two of the three ε's, A, B', and C'. The first is to contract A and C' (and A' and C), in which case the result does contain one baryon–antibaryon pair and has no backtrackers. The second is to contract A and B' (and A' and C), in which case the result of course again contains a baryon–antibaryon pair, but now has three backtrackers. In the Wilson case the first possibility therefore yields a term that contributes to the baryon propagator in the saddle point approximation, whereas in the second case the resulting term is zero. Fortunately these ambiguous terms are small, cf. Section 4, which supports our use of the saddle point approximation.

2. TECHNICALITIES

In this section we describe the calculational scheme that leads from a W-bank (a term in the expansion of the free energy) to a series of terms of the eventual effective action. The structure of a W-bank was explained in [1] and the SCELFE manual [7].

A detailed structure of the calculation and description of the computer algorithms will be given in a second manual [8]. Here we will only discuss the global structure.

The geometrical structure of a *W*-bank is given by a small number of sites, at most $2k + 2$ in a term of order β^k.

In a *W*-bank the product of a number of derivatives (1.13) occurs. For each such derivative of a one-link integral a series of the form (1.14) has to be substituted. The calculation thus splits up into a number of stages, or levels. On a normal level ("normal" to distinguish it from mu- and epsilon-levels, described below) a term from the series is selected. At this stage of the calculation color indices are made explicit in contrast to the situation in the *W*-banks, where they were implicit. Only the structure of the series (1.14) is kept in central memory. So after choosing one term from the series the assignment of the right color indices and the sites at the end of the link still has to be made.

The normal levels are numbered from 1 up, the maximum number being $3k + 1$ in order β^k, which is the maximum number of independent links in that order. Partial products of terms of level 1 to l are also stored at level l.

Derivatives (1.13) are needed with $0 < m + n \leqslant k_{\max}$, where $k_{\max}$ is the highest order considered in β. In the order to explain the color index substitution we first recall how such a derivative was coded in a *W*-bank. This was done by giving a number of pairs of integers $(p, g) = $ (plaquette number, type), where the type g defines the link of plaquette p as in Fig. 2. These pairs refer to the same actual link, the link on which the W_L to be differentiated resides. Furthermore a type 1 or 2 denotes differentiation w.r.t. $\bar{A}$ and type 3 or 4 differentiation w.r.t. A. A unique color index assignment can now be made as follows. If $a, b, c,$ and d denote the color indices attached to the sites of plaquette number p as indicated in Fig. 3, we make the substitution

$$(a, b, c, d) \rightarrow (a_{4p-3}, a_{4p-2}, a_{4p-1}, a_{4p}). \tag{2.1}$$

Defining the cyclic permutation η by

$$\eta: (1, 2, 3, 4) \rightarrow (4, 1, 2, 3), \tag{2.2}$$

this implies that the pair (p, g) gives rise to the color index pair $(a_{4p-4+g}, a_{4p-4+\eta g})$.
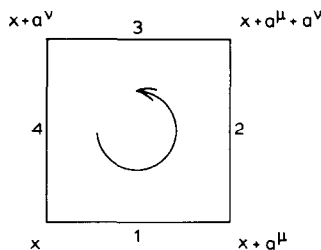

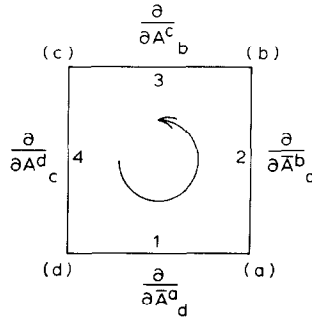FIG. 2. Numbering of the links of an oriented plaquette, $g = 1,...,4$.

Fig. 3.   Color indices and differentiation type corresponding to a link of an oriented plaquette.

The first index of this last pair corresponds to an index a_{2l+1} in (1.13) and will become a lower index in the result of the differentiation, and similarly the second index will correspond to an a_{2l+2} and become an upper index. For example, suppose that in the derivative needed at some place there are in total m derivatives w.r.t. A. If the ith pair (p, g) of this derivative in the W-bank with $g \geqslant 3$ is (p', g'), a_{2i-1} in (1.13) is to be replaced by $a_{4p'-4+g'}$ and a_{2i} by $a_{4p'-4+\eta g'}$. Similarly, if the jth pair with $g \leqslant 2$ is (p'', g''), $a_{2(m+j)-1}$ has to be replaced by $a_{4p''-4+g''}$ and $a_{2(m+j)}$ by $a_{4p''-4+\eta g''}$.

In the series (1.14) there are still some color indices implicit, for example, p and q in $\varepsilon^{abp}\varepsilon_{cdq}A^q_p$, because this expression is denoted by a type number and the external indices $a, b, c,$ and d only, see [8]. For these dummy indices also an explicit index is substituted.

When at a certain level a specific term has been singled out and the color index assignment and the assignment of the sites of the link have been made, the product of this term is made with the partial product of the terms up to the last level. Such a partial product consists of deltas, epsilons, and radicals. Making the product means that deltas are eliminated if one of the indices occurs elsewhere and the radical with common color indices are combined according to the rule (1.19). Two epsilon symbols with more than one index in common can also be directly eliminated, yielding a number or a delta symbol.

For the gauge group $U(3)$ with the choice κ, λ (1.10) and $\mathrm{tr}(A\bar{A})^3$ for the invariants, the last normal level would contain the end result, the contribution to the effective action. For $SU(3)$ some more steps are necessary that are described below. Up to now, instead of $\mathrm{tr}(A\bar{A})^3$ we have used the product $\mu\bar{\mu}$ as the third invariant, because the $U(3)$ results are then easily derivable from the $SU(3)$ results. But to calculate $U(3)$ up to higher order in β it is simpler to use $\mathrm{tr}(A\bar{A})^3$.

In case of $SU(3)$ a subseries g_i in (1.25) has to be substituted on each link. This is done at the next level, the mu level. The choice of g_i is indicated by labelling the derivatives of the one-link free energy w.r.t. the invariants with the number of uncompensated μ's added and the number of uncompensated $\bar{\mu}$'s. According to the restrictions given in Section 3 these extra labels can be $(0,0)$, $(1,0)$, $(2,0)$, $(0,1)$, or

(0, 2). The product at the highest normal level is then multiplied by the uncompensated μ's and $\bar{\mu}$'s at this mu-level.

If two epsilon symbols have only one index in common or are attached to the same point and have none in common, the result of contraction gives a sum of two, respectively, six terms. Therefore on top of the normal level and mu level, so-called epsilon levels are made, one for each pair of singly or not-contracted (but contractable) epsilons. At each of these epsilon levels again a choice has to be made from two or six possibilities. The number of these epsilon levels lies between 0 and $4k$ in order β^k.

The product of the highest epsilon level is the contribution to the effective action. The baryon and meson content is coded in a form corresponding to (1.20) and (1.22). For (1.22) this is done by giving the sequence $(p, r_1, ..., r_k)$ and changing the sign of the overall coefficient. For (1.20) the site $q \equiv (s_i)_0$ in which the antibaryon sits, the site $p \equiv (s_i)_{k_i+1}$ where the baryon is located and the three (possibly empty) strings s_i between q and p are given. More complicated baryon terms are discarded up to now, cf. Section 3.

After a contribution to the effective action has thus been completed, it is compared to terms previously generated and put in a buffer. If it has the same structure as one already present, its coefficient is added to that in the buffer, otherwise it is added at the end of the buffer. If the buffer is full it is written to a disk. Later a comparison between terms in different buffers still has to be performed in order to add terms of the same structure.

3. IMPLEMENTATION AND RESULTS

First we discuss some restrictions that are made at present in the implementation.

In order to be able to calculate the baryon propagator and baryon–meson coupling and scattering amplitudes we retain terms with only one baryon–antibaryon pair (with the baryon and antibaryon at different sites), but discard terms containing two or more of such pairs.

Also a restriction is made regarding the number of uncompensated μ's and $\bar{\mu}$'s, cf. (1.25). As the number of those μ's in a term increases, the contribution from such a term decreases quite rapidly (cf. Section 4). Furthermore, at $\beta = 0$ contributions of order $(\mu\bar{\mu})^2$ may already be neglected because of their smallness (in the Wilson case they vanish). In general, the addition of many μ's leads either to many baryons (and those terms are discarded) or to baryon loops. In Appendix A an example is given with four μ's. Such terms turn out to be small. The remaining possibility is that the μ's form a baryon–antibaryon combination with a long string of mesons in between. On the two plaquettes of the second order calculation for $SU(3)$ each two sites can be connected by a string of three links and corrections due to longer strings from long series of μ's are down by factors comparable to the one calculated in Appendix A. Therefore, terms with more than three uncompensated μ or $\bar{\mu}$'s are neglected. Furthermore, if a term μ^3 or $\bar{\mu}^3$ in (1.25) is used, many baryons arise of which most have to be compensated in order to end up with at most one baryon–antibaryon pair.

This compensation requires that the diagram without those μ's already contains at least two baryons and antibaryons. The contributions of such terms are very small and are therefore neglected. To summarize, we keep terms that have:

   (i)   no more than 3 uncompensated μ and $\bar{\mu}$'s in total, and

   (ii) no more than 2 uncompensated μ or $\bar{\mu}$'s on any one link.

The programs to implement the calculational scheme of Section 2 were written mostly in Fortran and partly in the CDC assembler language Compass. They make extensive use of the 60 bit word structure of the CDC machines and of the shift and mask instructions available to manipulate parts of these words. The programs were executed on a CDC-cyber 173 with operating systems NOS/BE level 538 of NIKHEF-H [9] and a CDC-cyber 170-750 with operating system NOS/BE level 552 of SARA [10].

In first order in β there is only one W-bank that leads to 3932 terms of the effective action and requires 1979 CPU-sec on the cyber 173, if any number of backtrackers are allowed (but only one baryon and antibaryon). Of the 54 terms in second order in β one needs only to process 12, as the others can be obtained from these in a simple fashion. Until April 1983 only terms with at most 4 backtrackers and one baryon and antibaryon had been calculated. These calculations require about 45 CPU-h/term on the CY 173 and result in about 85,000 contributing terms per incoming term.

4. THE EFFECTIVE POTENTIAL

As an application of the effective action, the effective potential will be calculated in this section. We will restrict ourselves to Wilson fermions ($r = 1$). The $U(3)$ effective potential is obtained by inserting the vacuum expectation value for the meson field in the effective action [2, 3]

$$V = -S(\mathcal{M} = v\mathbb{1})/\text{vol}. \tag{4.1}$$

The effective potential can be expressed as

$$V = V_0(M, v) + \beta V_1(v) + \frac{\beta^2}{2} V_2(v) + \cdots, \tag{4.2}$$

where, for N_f flavours of equal mass,

$$V_0 = -4NN_f(\ln v - Mv). \tag{4.3}$$

In (4.2) V_1 and V_2 are functions of v that will be calculated below.

The extremum of the effective potential determines one relation between M and v, and fitting the pion mass leads to another relation. Together these determine the β-

dependence of M and v. For vanishing pion mass the expansion of M and v start off as [2, 3]

$$v = \frac{1}{2} + v_1\beta + v_2\frac{\beta^2}{2} + \cdots, \tag{4.4a}$$

$$M = M_c = 2 + M_1\beta + M_2\frac{\beta^2}{2} + \cdots. \tag{4.4b}$$

Here M_c is the critical value for the mass parameter, for which the pion mass vanishes.

Here we shall only be concerned with the determination of V_1 and V_2. To calculate these from the effective action one needs to calculate traces of products of the projectors $P_\mu^\pm$. This can be done quite elegantly with a method devised by Stamatescu [11], with a computer program written by Lang [12]. The results can be summarized as

$$V_1 = -NN_f\frac{1}{8}(2v)^4, \tag{4.5a}$$

$$V_2 = \frac{-4}{3}N^2 - NN_f\left\{\frac{1}{8}(2v)^6 + \frac{5055}{2^{18}}(2v)^8\right. \tag{4.5b}$$

$$\left. + \frac{29865}{2^{20}}(2v)^8 N_f + \frac{531441}{2^{26}\cdot 10^4}(2v)^{16}(N_f + 4N_f^2 + 4N_f^3)\right\}.$$

Note that $v = \frac{1}{2}$ at $\beta = 0$.

The terms of order v^{16} in (4.5b) are spurious in the sense that they would not have occurred in the effective potential if the invariant $\text{tr}(A\bar{A})^3$ had been chosen instead of $\det A * \det \bar{A}$ (recall the discussion in Section 1). Therefore, they must be small in order for this approach to make sense. From Table I it can be seen that this is indeed the case even for $N_f = 4$. As the effective expansion parameter is proportional to N_f/N the case of many flavours should not be taken too seriously.

TABLE I

Coefficients of Different Powers of $(2v)$ in $-8V_2/N N_f$ for $U(3)$

N_f	$(2v)^6$	$(2v)^8$	$(2v)^{16}$	$^0/_{00}{}^a$
1	1.0000	.3821	.0001	.04
2	1.0000	.6100	.0003	.2
3	1.0000	.8378	.0009	.5
4	1.0000	1.0657	.0021	1.0

[a] In the last column the promillage of the 16th order terms w.r.t. the sum of the 6th and 8th order terms is given.

Noteworthy is the size of the $O(v^8)$ contributions, which for two flavours is some 60% of that of $O(v^6)$.

To make contact with the hopping parameter expansion [13, 14] the mass parameter M should be replaced by $1/2K$ and the fields rescaled to make the coefficient of the mass term equal to one. For the present formulation this would mean that a factor K comes with each projector $P^\pm$. For the contributions to the potential (closed loops) a factor K would come with each v. So the order in v is equal to the order in the hopping parameter K. The hopping parameter expansion is known to converge slowly [12] and this is substantiated by the above results for $U(3)$ and the results below for $SU(3)$.

For $SU(3)$ the loop expansion method developed in [2] leads, after integrating over the baryon fields, to an effective meson action

$$S_{eff}(\mathcal{M}) = S(\mathcal{M}) + \lambda S_B(\mathcal{M}) + O(\lambda^2). \tag{4.6}$$

The effect of S_B is rather small. It is a term due to hopping of the meson field dependent baryon propagator. It will be neglected here, hence everything is to $O(\lambda^0)$. Expressing the effective potential again in the form (4.2) now leads to the result that V_0 is again given by (4.3) and V_1 by

$$V_1 = -NN_f \left\{ \frac{1}{8}(2v)^4 + \frac{81}{16384}(2v)^8 + \frac{81}{8192}(2v)^8 \right\}. \tag{4.7}$$

Before giving V_2, we introduce a parameter that counts the number of uncompensated μ or $\bar\mu$'s in the contribution. Let ξ^p denote p uncompensated μ or $\bar\mu$'s. The numerical value of ξ is 1. Then V_2 is given by

$$V_2 = -\frac{4}{3}N^2 - NN_f \left\{ \frac{1}{8}(2v)^4 + \frac{1}{8}(2v)^6 \right.$$

$$+ \frac{5055}{2^{18}}(2v)^8 + \frac{29865}{2^{20}}(2v)^8 N_f + \frac{81}{2^{12}}(2v)^{10}$$

$$+ \frac{81}{2^{11}}(2v)^{10} N_f + \frac{81}{2^{25}}(2v)^{12}[768 - 694\xi + 81\xi^2 - 54\xi^3]$$

$$+ \frac{243}{2^{26}}(2v)^{12} N_f[-781 - 1388\xi + 162\xi^2 - 108\xi^3]$$

$$+ \frac{81}{2^{25}}(2v)^{12} N_f^2[-5415 - 1388\xi + 162\xi^2 - 108\xi^3]$$

$$\left. + \frac{2187}{2^{24}}(2v)^{14}[1 + 4N_f + 4N_f^2] + O(v^{16}) \right\}. \tag{4.8}$$

TABLE II

Fermion Contributions to $-8V_2/N\,N_f$ for $SU(3)$ and $N_f = 2$

Coeff. of $(2v)^p$	ζ^0	ζ	ζ^2	ζ^3
4	1.000			
6	1.000			
8	.610			
10	.791			
12	−.449	−.201	.023	−.016
14	.026			
16	−.048	−.060	.005	−.004
18				.001
20				−.0004
22				.003
24				.001

These results are displayed in another way for two flavours in Table II. With the restrictions defined in Section 3 terms up to order v^{24} appear. From order v^{12} on contributions of order ζ^4 and higher do exist but have been neglected as explained in Section 3. Both from (4.8) and Table II one can see that coefficients of higher orders in ζ become small. Also it can be seen that terms with possible ambiguities (a subset of the terms or order ζ^2) are rather small, cf. Section 2. The highest order in the hopping parameter for which the contributions are still considerable is 12 and it looks worthwhile to keep terms up to first order in ζ, certainly if one wants to go up to 3 flavours.

5. Conclusions

The effective action has been calculated for $SU(3)$ up to order β^2. It seems hard, due to the amount of computer time needed, to push the series beyond second order. It might be possible to do the third order, if one restricts oneself to order ζ. For $U(2)$ and $U(3)$ the prospects are better and $U(1)$ can certainly be done up to order β^4.

The sample calculation of the effective potential discussed in Section 4 shows that the $O(v^8)$ contributions to the $U(3)$ effective potential are not small compared to the $O(v^6)$. For the $SU(3)$-potential in second order $O(v^{12})$ contributions are still not negligible, whereas there the leading fermionic contribution is $O(v^4)$.

Other results obtained with the effective action calculated as described here will be published elsewhere [3].
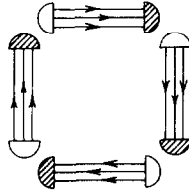
FIG. 4. Configuration of four $\mu/\bar{\mu}$'s giving rise to corrections of, e.g., the effective potential. The open semicircle denotes an anti-baryon, the full semicircle a baryon, and the line segments $P^{\pm}$'s.

APPENDIX A

The contributions of terms with many μ's are suppressed with respect to comparable contributions without these μ's because of the smallness of the coefficients in g_1, with respect to the coefficient of the same combination of invariants in g_0. Let us, for example, consider the configuration depicted in Fig. 1. We compare the value of the simplest contribution in order β of the $SU(3)$ effective potential (the $O(v^4)$ term in (4.7)) which has g_0's on every link, to the same contribution with g_1's on every link. Replacing the g_0's by g_1's results in changing the coefficients on each link by a factor, $\frac{1}{24}$ in this case. Furthermore, the result has to be multiplied by the value of the μ-configuration of Fig.4, which is (after contracting the epsilons and substituting $\mathcal{M} = v\mathbb{1}$),

$$\frac{(3v)^{12}}{96} N_f(1 + 3N_f + 2N_f^2).\tag{A.1}$$

Using the value $v = \frac{1}{2}$ which is valid at $\beta = 0$ and 3 flavours, this equals 114. The factor from the changed coefficients is $(\frac{1}{24})^4$, resulting in a total suppression factor of 3.42×10^{-4}, which is very small.

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